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Statistics of reduced words in locally free and braid groups: Abstract studies and applications to ballistic growth model

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Abstract. We study numerically and analytically the average length of reduced (primitive) words in the so-called locally free ($\mathcal{LF}_n(d)$) and braid (B_n) groups. We consider the situations when the letters in the initial words are drawn either without or with correlations. In the latter case, we show that the average length of the reduced word can be increased or lowered depending on the type of correlation. The ideas developed are used for analytical computation of the average number of peaks of the surface appearing in some specific ballistic growth model.

Introduction

This paper is devoted to the elaboration of a general method of analysis of stable probability distributions in statistical systems of completely different physical nature, such as: vortices in superconductors, entangled polymer bunches and open surfaces of growing media. Our investigation is forced by realizing the following facts.

(a) The nematic-type ordering in bunches of entangled polymers as well as the consideration of thermodynamic properties of uncrossible vortex lines immediately turn us to studying the statistics of mutually entangled chain-like objects with braid-like topology.

(b) The ballistic-type growth of some amount of deposit in a box and the investigation of the surface shape can be easily translated into the language of random walks over the elements of some noncommutative group.

The development of mathematical methods implies the construction of the statistical theory of random walks on noncommutative groups [1–10]; while the application of elaborated methods in physics is aimed to answer the following question [11]: how does the change in the topological state of the system affects its physical properties?

Although the general concepts of the noncommutative probability theory have been well elaborated in the field-theoretic context, their application in the related areas of mathematics and physics, such as, for instance, statistical physics of chain-like objects is highly limited. This state of affairs can be accounted for by two facts: (a) there is a communication problem, i.e. the languages used by specialists in topological field theory and probability theory are completely different at first glance; (b) physical systems give no evidence how these ideas are reflected in simple geometrical examples.

This paper is mainly concerned with the probabilistic methods which allow us to solve the basic problems dealing with the limit distributions of random walks on some

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of the simplest noncommutative groups. To be more specific, we consider analytically and numerically the limit behaviour of the Markov chains where the states are randomly taken from some noncommutative discrete group. In particular, we restrict ourselves with the so-called *braid* (B_{n+1}) and *locally free* ($\mathcal{LF}_{n+1}$) groups—see the definitions in section 1.2. The preliminary results concerning the words' statistics in locally free groups appeared in recent works [4, 10].

The paper is organized as follows: in section 1 we introduce the necessary definitions; sections 2 and 3 are devoted to the consideration of random walks on the locally free and braid groups without and with correlations in the word writings; the application of the concept of locally free groups to the problem of ballistic growth is considered in the section 4.

1. Basic definitions

We begin with the investigation of the probabilistic properties of Markov chains on simplest noncommutative groups. In the most general way the problem can be formulated as follows (see also [4]).

Take a discrete group $\mathcal{G}_{n+1}$ constructed by the finite number of generators $\{g_1, \ldots, g_n\}$. Any arbitrary sequence of generators we call the *initial word*. The *length*, N, of this word is the total number of used generators ('letters'), whereas the *length* of the *reduced* (or *primitive*) *word*, μ, is the number of letters which remain after applying of all possible group relations.

Later on we mainly use the rescaled variables $N' \equiv N/n$ and $\mu' \equiv \mu/n$ instead of N and μ and consider the situation $n \gg 1$ neglecting the 'edge effects'.

In the following the most attention is paid to the computation of the mean length, $\langle \mu'(N') \rangle$, averaged over various distributions of initial words belonging to the group $\mathcal{G}_{n+1}$ ($\mathcal{G}_{n+1}$ is either a 'locally free' or braid group).

1.1. Random walks over group elements

Take the group $\mathcal{G}_{n+1}$. Let p be some distribution on the set $\{g_1, \ldots, g_n, g_1^{-1}, \ldots, g_n^{-1}\}$. For convenience we call $h_j \equiv g_i$ for $j = i$ and $h_j \equiv g_i^{-1}$ for $j \equiv i + n$. We construct the (right-hand) random walk (the random word) on $\mathcal{G}_{n+1}$ with a transition measure, p, i.e. we add, with the probability p, the element $h_{\alpha_{N+1}}$ to the given word $W_N = h_{\alpha_1} h_{\alpha_2} \ldots h_{\alpha_N}$ from the right-hand side†.

The random word W formed by N letters taken with the probability distribution p from the set $\{g_1, \ldots, g_n, g_1^{-1}, \ldots, g_n^{-1}\}$ is called the *initial word* of the length N on the group $\mathcal{G}_{n+1}$.

We distinguish below between the following three situations.

Drawing words without any correlations ('standard case'). The probability distribution p is uniform, i.e. $p = 1/2n$ on the set $\{g_1, \ldots, g_n, g_1^{-1}, \ldots, g_n^{-1}\}$.

Besides this standard case, we consider also two extreme situations of a words construction, hereafter refered as 'weak (strong) correlations'.

† Analogously we can construct the left-hand side random walk on the group $\mathcal{G}_{n+1}$.

Drawing words with weak correlations (regime 'A'). Suppose that in the initial word, W_N, we have for the last letter $h_{\alpha_N} = g_k$ or g_k^{-1}. Then we add the next $(N+1)$th letter $h_{\alpha_{N+1}}$ with the following probabilities:

$$h_{\alpha_{N+1}} = \begin{cases} g_k^{\pm 1} & \text{with probability } q_A \\ \text{any other letter} & \text{with probability } p_A. \end{cases} \tag{1}$$

The normalization reads,

$$2q_A + 2(n-1)p_A = 1. \tag{2}$$

Drawing words with strong correlations (regime 'B'). Suppose again that in the initial word, W_N, we have for the last letter $h_{\alpha_N} = g_k$ or g_k^{-1}. Then we add the next $(N+1)$th letter $h_{\alpha_{N+1}}$ with the following probabilities:

$$h_{\alpha_{N+1}} = \begin{cases} g_{k\pm 1}^{\pm 1} & \text{with the probability } q_B \\ \text{any other letter} & \text{with the probability } p_B. \end{cases} \tag{3}$$

The normalization in this case reads,

$$4q_B + 2(n-2)p_B = 1. \tag{4}$$

In particular, we show below that in the case 'A' the length of the reduced (primitive) word decreases when q_A is increased, while in the case 'B' the length of the reduced word increases when q_B is increased.

The absence of correlations, in both cases 'A' and 'B', means setting $q_{A,B} = p_{A,B} = 1/2n$. Thus, in the limit $n \gg 1$, the 'standard case' is recovered, formally, by setting $q_{A,B} = 0$ in the equations.

The investigation of such correlations is necessary in view of future physical applications [11], especially when we are dealing with the polymers entanglements. Indeed, if we think of $\mathcal{G}_{n+1}$ as of a braid group, the weak ('A') and strong ('B') correlation regimes will correspond, respectively, to the weak and strong 'entanglement regimes'.

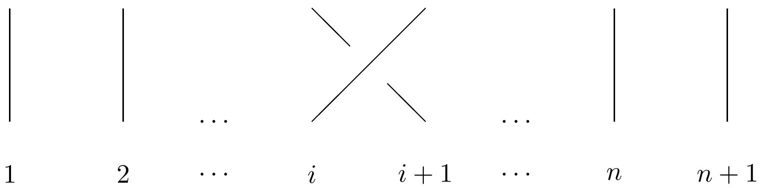
1.2. Braid and 'locally free' groups

We aim to study the asymptotics of the limit distributions of Markov chains on the braid group B_{n+1}. For the case $n = 2$ the problem has been solved in [4], where the limit probability distribution as well as the conditional limit probability distribution of 'Brownian bridges' on the group B_3 has been derived. For $n > 2$ this problem is as yet unsolved. However, we can extract some reliable estimations for the limit behaviour of Markov chains on B_{n+1} considering the random walks on so-called 'locally free groups' [2, 4, 10].

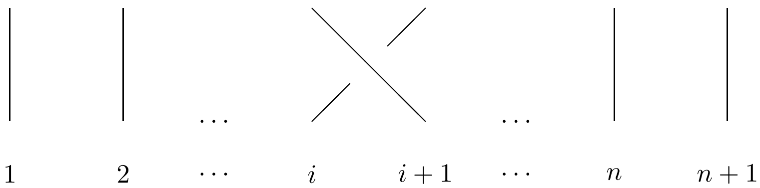
Braid group. The braid group B_{n+1} of $n+1$ strings has n generators $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ with the following relations:

$$\begin{aligned} &\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad (1 \leqslant i < n) \\ &\sigma_i\sigma_j = \sigma_j\sigma_i \qquad (|i-j| \geqslant 2) \\ &\sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = \hat{e}. \end{aligned} \tag{5}$$

• The word written in terms of 'letters'—generators from the set $\{\sigma_1, \ldots, \sigma_n, \sigma_1^{-1}, \ldots, \sigma_n^{-1}\}$—gives a particular *braid*. Schematically, the generators σ_i and σ_i^{-1} could be represented as follows:



$= \sigma_i$



$= \sigma_i^{-1}$.

• Diagramatically, the braid can be represented as a set of crossed strings going from the top to the bottom after 'gluing' the braid generators.

• The closed braid appears after gluing the 'upper' and the 'lower' free ends of the braid on the cylinder.

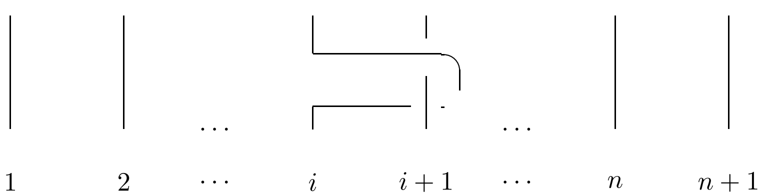
• Any braid corresponds to some knot or link. So, there is a strong possibility of using the braid group representation for the construction of topological invariants of knots and links, but the correspondence of braids and knots is not mutually single valued and each knot or link can be represented by an infinite series of different braids.

Locally free group. The group $\mathcal{LF}_{n+1}(d)$ is called *locally free* if the generators, $\{\sigma_1, \ldots, \sigma_n\}$ obey the following commutation relations:

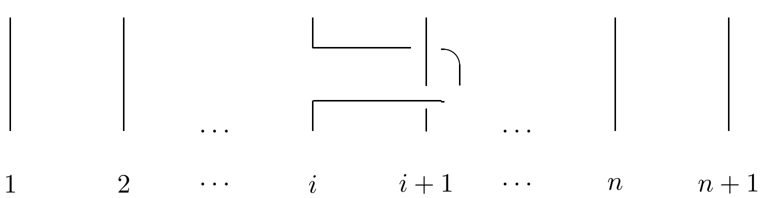
(a) each pair (σ_j, σ_k) generates the free subgroup of the group $\mathcal{LF}_{n+1}$ if $|j - k| < d$;

(b) $\sigma_j \sigma_k = \sigma_k \sigma_j$ for $|j - k| \geqslant d$.

We will be concerned mostly with the case $d = 2$ for which we define $\mathcal{LF}_{n+1}(2) \equiv \mathcal{LF}_{n+1}$. The graphical representation of generators σ_i and σ_i^{-1} is rather similar to that of the braid group:



$= \sigma_i$



$= \sigma_i^{-1}$.

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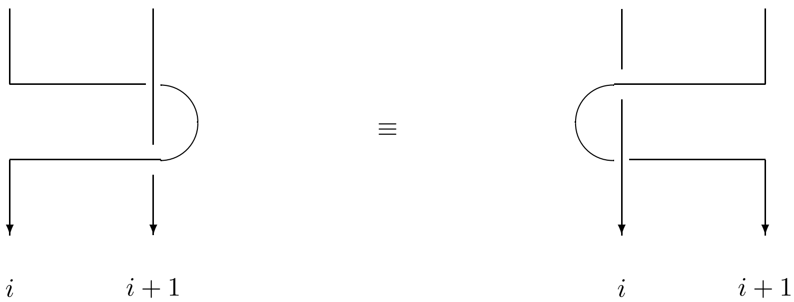
It is easy to understand that the following geometrical identity is valid:



hence, it is unnecessary to distinguish between 'left' and 'right' operators σ_i.

It can be seen that the only difference between the braid and locally free groups consists of elimination of the Yang–Baxter relations (first line in equation (5)).

2. Random walks without correlations on locally free and braid groups

It has been shown in papers [1, 6–8] that for the free group (i.e. for the group without any commutation relations among generators) the problem of the limit distribution of Markov chains can be mapped to the investigation of statistics of random walks on a simply connected tree. In the case of locally free groups or braid groups the more complicated structure does not allow us to use this simple geometrical image directly. Nevertheless, we show in the following that simple ideas allow us to treat rigorously the locally free case. Moreover, this approach will be especially useful when we study braid groups. Including the braiding relations (5) in only a probabilistic way, we will obtain, for this latter case, approximate results. Comparison with numerical simulations will show, however, that the approximation is a rather good one.

2.1. *The locally free group* $\mathcal{LF}_{n+1}(2)$

Let us begin with the following example.

Example 1. Suppose that the N-letter initial word leads to the following reduced word:

$$\sigma_1^{-1}\sigma_2\sigma_1\sigma_4\sigma_2^{-1}\sigma_7\sigma_3\sigma_5^{-1}\sigma_3\sigma_8^{-1}.$$

Now, if we randomly add a new letter from the right-hand side, it is easy to see that only σ_3, σ_5^{-1} or σ_8^{-1} can be reduced (for instance, σ_7 cannot be reduced even if, by chance, we add σ_7^{-1} because this generator cannot pass through $\sigma_8^{-1}, \ldots$).

Definition 1. The set of letters which we can reduce in the given primitive word by adding one extra letter from the right-hand side we call *the set of reducible letters*, I.

The number of letters belonging to I we denote as η.

In the above example $I = \{\sigma_3, \sigma_5^{-1}, \sigma_8^{-1}\}$ and $\eta = 3$. Generally speaking, $\eta' \equiv \eta/n$ is a random variable, the probability distribution of which *a priori* depends both on $N' \equiv N/n$ and n.

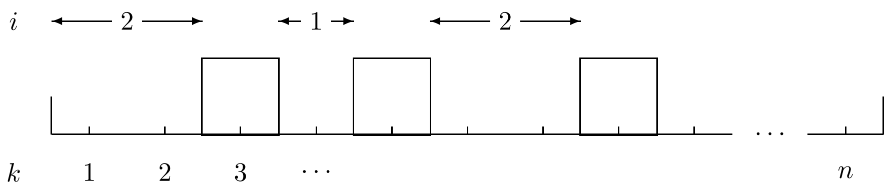
It is worth mentioning the following basic properties of the set I:

(i) if $\sigma_i^{\pm 1}$ belongs to I then $\sigma_i^{\mp 1}$ does not belong to I;

(ii) if $\sigma_i^{\pm 1}$ belongs to I then $\sigma_{i+1}^{\pm 1}$, $\sigma_{i+1}^{\mp 1}$, $\sigma_{i-1}^{\pm 1}$ and $\sigma_{i-1}^{\mp 1}$ do not belong to I, i.e. *all the elements of I must commute*.

On the basis of (i) and (ii) we can easily deduce that $0 \leqslant \eta \leqslant n/2$ ($\eta = 0$ corresponds to a completely reduced word, i.e. $\mu = 0$).

The set I allows the following very useful geometrical interpretation. Take n boxes (labelled as $k = 1, \ldots, n$) as displayed below:



The box k is empty except if $\sigma_k^{\pm 1}$ belongs to I. In the given example only boxes 3, 5, 8 are occupied. From the properties (i) and (ii) of the set I we deduce that two neighbouring boxes cannot be occupied.

Generally, I is described by occupied boxes separated by a sequence of i ($i \geqslant 1$) empty boxes. Let n_i be the number of such sequences of length i. Neglecting the edge effects (i.e. for $n \gg 1$), we obtain the following rules:

$$\sum_{i \geqslant 1} n_i = \eta \tag{6}$$

and

$$\sum_{i \geqslant 1} i\ n_i = n - \eta. \tag{7}$$

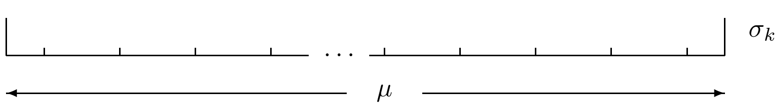
Consider now the evolution of the reduced word (length μ) and of the set I (length η) when we randomly add a letter from the right-hand side: $N \to N+1$ (i.e. $N' \to N' + 1/n$). (Apparently the evolution of the reduced word and the set I are correlated.) Two possibilities can occur:

$$\begin{cases} \Delta\mu' = +\dfrac{1}{n} & \text{for the 'increase' process } (\mu \to \mu + 1) \\ \Delta\mu' = -\dfrac{1}{n} & \text{for the 'decrease' process } (\mu \to \mu - 1) \end{cases}$$

where $\Delta\mu'$ stands for the increment $\mu'(N' + 1/n) - \mu'(N')$.

We consider the 'increase' and 'decrease' processes separately.

The 'increase' process.



It is easy to see that the added letter will necessarily belong to the new set I. However, it does not mean that, in this case, we *automatically* have $\eta' \to \eta' + 1/n$. Actually, η' can stay unchanged or changed by $\pm 1/n$.

The latter point becomes clearer if we come back to example 1 where $I = \{\sigma_3, \sigma_5^{-1}, \sigma_8^{-1}\}$. We have the following choices:

- if we add σ_3 (or σ_5^{-1} or σ_8^{-1}), then the set I (and, hence, η') remains to be unchanged;
- if we add σ_6, then η' is still unchanged: I becomes the new set $\{\sigma_3, \sigma_6, \sigma_8^{-1}\}$, the letter σ_6 has replaced σ_5^{-1} in the set I. The same occurs for η' if, instead of σ_6, we add σ_6^{-1} or $\sigma_7^{\pm 1}$, and so on...;

• if we add σ_4, then I becomes the new set $\{\sigma_4, \sigma_8^{-1}\}$ (σ_4 erases σ_3 and σ_5^{-1} from I) and, consequently, $\eta' \to \eta' - 1/n$ (same change occurs for η' if we add σ_4^{-1});

• if we add σ_{10}, then I becomes $\{\sigma_3, \sigma_5^{-1}, \sigma_8^{-1}, \sigma_{10}\}$ and $\eta' \to \eta' + 1/n$ (the same is happened if we add $\sigma_1^{\pm 1}$ or σ_{10}^{-1}, and so on...).

These considerations can be generalized and careful inspection leads to the following rules for the increasing process ($\Delta N' = +1/n$, $\Delta\mu' = +1/n$):

$$\begin{cases} \Delta\eta' = 0 & \text{occurs with probability } \Pi_0 = \frac{1}{2n}\left(\eta + 4\sum_{i\geqslant 2} n_i\right) \\ \Delta\eta' = -\dfrac{1}{n} & \text{occurs with probability } \Pi_- = \frac{1}{2n}\, 2n_1 \\ \Delta\eta' = +\dfrac{1}{n} & \text{occurs with probability } \Pi_+ = \frac{1}{2n} 2\sum_{i\geqslant 3} n_i(i-2). \end{cases} \tag{8}$$

With the help of equations (8) we obtain as expected, for the total probability, Π_1, of the increasing process,

$$\Pi_1 = \Pi_0 + \Pi_- + \Pi_+ = 1 - \frac{\eta'}{2}. \tag{9}$$

From the inequality $0 \leqslant \eta' \leqslant \frac{1}{2}$ derived above it immediately follows that $\frac{3}{4} \leqslant \Pi_1 \leqslant 1$. For the corresponding average change of η' we have,

$$\frac{\langle \Delta_1 \eta' \rangle}{\Delta N'} = -\frac{n_1}{n} + \frac{1}{n}\sum_{i\geqslant 3} n_i(i-2) = \frac{1}{n}\sum_{i\geqslant 1} n_i(i-2) = \frac{n - 3\eta}{n}$$

so, we arrive at the following equation

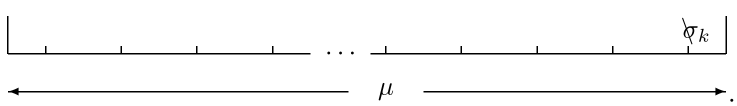
$$\langle \Delta_1 \eta' \rangle = (1 - 3\langle \eta' \rangle)\Delta N' \tag{10}$$

where $\langle \cdots \rangle$ represents an average over the set of all initial words with N' fixed.

The 'decrease' process. Now we compute the change $\langle \Delta_2 \eta' \rangle$ for the reducing process, i.e. when $N' \to N' + 1/n$ and $\mu' \to \mu' - 1/n$. It occurs with the probability

$$\Pi_2 \equiv 1 - \Pi_1 = \frac{\eta'}{2}.$$

In this operation, a letter of the set I is erased and, again, we have $\Delta\eta' = 0$ or $\pm 1/n$. Recall that all the elements of I commute. So, the erased letter (here σ_k) can always been considered as the last one:



From this point of view, the decrease process ($N \to N+1$, $\mu \to \mu - 1$) is rigorously the inverse of the increase one ($N-1 \to N$, $\mu - 1 \to \mu$). Thus, weighting each process with its actual probability, we obtain the equation,

$$\langle \Delta_2 \eta' \rangle = -\left\langle \Delta_1 \eta' \frac{\Pi_2}{\Pi_1} \right\rangle \tag{11}$$

where corrections of order $1/n$ are neglected.

Collecting the 'increase' and 'decrease' processes together, we obtain,

$$\frac{\langle \Delta \eta' \rangle}{\Delta N'} = \left\langle (1 - 3\eta')\left(1 - \frac{\Pi_2}{\Pi_1}\right) \right\rangle$$

and

$$\frac{\langle \Delta\mu' \rangle}{\Delta N'} = \Pi_1 - \Pi_2.$$

We arrive in the limit $N \gg 1$; $\mu \gg 1$ at the following differential equations:

$$\frac{\mathrm{d}\langle \eta' \rangle}{\mathrm{d}N'} = \left\langle (1-3\eta')\frac{(1-\eta')}{(1-\eta'/2)} \right\rangle \tag{12}$$

and

$$\frac{\mathrm{d}\langle \mu' \rangle}{\mathrm{d}N'} = 1 - \langle \eta' \rangle. \tag{13}$$

It should be stressed that equation (13) together with the inequality $\eta' \leqslant \frac{1}{2}$ imply that $\langle \mu \rangle / N \geqslant \frac{1}{2}$.

We can discard the brackets in equations (12) and (13) when $n \gg 1$. To show that, let us compute the probability distribution of η', $P(N', \eta')$. The function $P(N', \eta')$ satisfies the following recursion relation,

$$\begin{cases} P(N'+1/n, \eta') = P_0\, P(N', \eta') + P_1\, P(N', \eta' - 1/n) + P_2\, P(N', \eta' + 1/n) \\ P(0, \eta') = \delta(\eta') \end{cases} \tag{14}$$

where

$$P_0 = \Pi_0 \left(1 + \frac{\Pi_2}{\Pi_1}\right) \qquad P_1 = \Pi_+ + \Pi_- \frac{\Pi_2}{\Pi_1} \qquad P_2 = \Pi_- + \Pi_+ \frac{\Pi_2}{\Pi_1}$$

are transition rates. Expanding equation (14) to the lowest order in $1/n$ we obtain

$$\frac{\partial P}{\partial N'} = (P_2 - P_1)\frac{\partial P}{\partial \eta'} + \frac{1}{2n}(P_1 + P_2)\frac{\partial^2 P}{\partial \eta'^2} + \mathrm{O}\left(\frac{1}{n}\right). \tag{15}$$

When $n \to \infty$, the diffusion term becomes negligible and the equation becomes deterministic. Then, the distribution function P acquires zeros width, hence η' is peaked at its average value. The same would be true for μ' but not for variables η and μ (for which a nonvanishing width is expected).

From now on, as far as only η' and μ' are concerned, we systematically omit the brackets. Solving equation (12) we get

$$\frac{1-\eta'}{(1-3\eta')^{5/3}} = \mathrm{e}^{4N'}. \tag{16}$$

Using equation (13), we obtain μ' as a function of N'.

The comparison with the numerical simulations is displayed in the upper part of figure 1 (the full curve: equations (13), (16); the points: simulations with $n = 100$).

We observe at small N' that $\langle \mu \rangle \simeq N$, i.e. practically no reduction occurs because the words are too short compared with the set of available letters and we have only a small chance to draw, in the same word, a given generator and its inverse. On the other hand, taking the limit $N' \gg 1$ in equations (13)–(16) we arrive at [10] :

$$\eta' = \frac{1}{3} \qquad \text{and} \qquad \frac{\langle \mu \rangle}{N} = \frac{2}{3}. \tag{17}$$

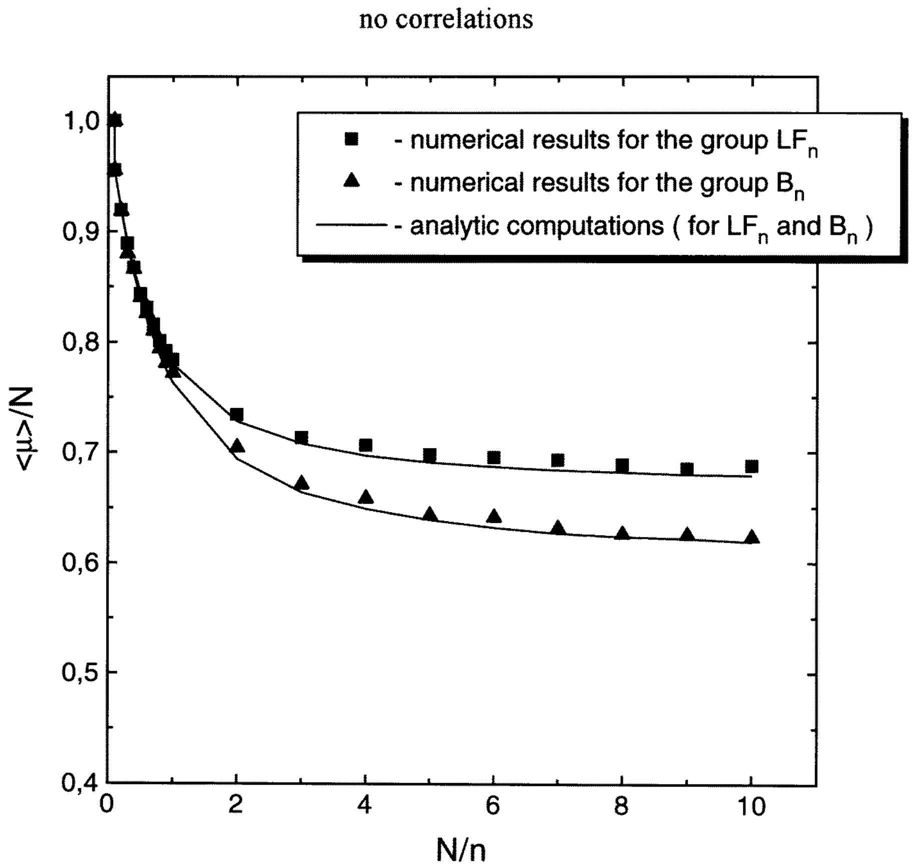


Figure 1. The normalized length, $\langle\mu\rangle/N$, of the reduced word as a function of the length of the initial word, N/n, for locally free and braid groups. Words are drawn without any correlations.

2.2. The locally free group $\mathcal{LF}_{n+1}(d)$ for $d \geqslant 2$

The ideas developed above can be extended to the general case—the group $\mathcal{LF}_{n+1}(d)$ with $d \geqslant 2$. It is worth mentioning the following simple fact. The generator σ_k erases all σ_j's with $j = k-(d-1), \ldots, k-1, k+1, \ldots, k+(d-1)$ from the set I. In other words, σ_k 'screens' all the generators in a zone of extension $2(d-1)$ around itself. This point of view is especially useful when we treat the correlations ('B').

In the case of the group $\mathcal{LF}_{n+1}(d)$ equations (8) become:

$$\begin{cases} \Delta\eta' = 0 & \text{occurs with } \Pi_0 = \frac{\eta'}{2} + \sum_{i=d-1}^{2(d-1)} \frac{4n_i(i-(d-1))}{2n} + \sum_{i>2(d-1)} \frac{4n_i(d-1)}{2n} \\ \Delta\eta' = +\dfrac{1}{n} & \text{occurs with } \Pi_+ = \sum_{i>2(d-1)} \frac{2n_i(i-2(d-1))}{2n} \\ \Delta\eta' = -\dfrac{1}{n} & \text{occurs with } \Pi_- = \sum_{i=d-1}^{2(d-1)} \frac{2n_i(2(d-1)-i)}{2n} \end{cases} \tag{18}$$

while equation (13) remains unchanged.

The solution of equation (12) now reads

$$\frac{1-\eta'}{(1-(2d-1)\eta')^{\frac{4d-3}{2d-1}}} = \mathrm{e}^{4(d-1)N'} \tag{19}$$

(cf equation (16)). Asymptotically we obtain:

$$\eta' = \frac{1}{2d-1} \qquad \text{and} \qquad \frac{\langle\mu\rangle}{N} = \frac{2d-2}{2d-1}. \tag{20}$$

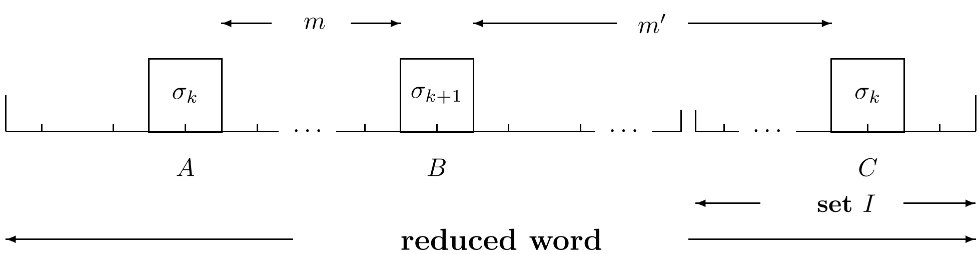
It is easy to check that equation (17) is recovered for $d = 2$ (see also [10]).

2.3. *The braid group B_{n+1}*

Comparing the groups B_{n+1} and $\mathcal{LF}_{n+1}(2)$ we can see that η' in equation (13) has to be replaced by some η'' ($> \eta'$) in order to take into account the additional braiding relations equation (5).

For each σ_k belonging to I, we can obtain additionnal reduction if and only if we can build at the end of the reduced word a sequence of letters like $\sigma_k\sigma_{k+1}\sigma_k$. Then the braiding relation equation (5) implies that σ_{k+1} becomes reducible.

Let us compute the probability Q of finding such a sequence. Suppose that generators σ_k and σ_{k+1} emerge elsewhere in the reduced word. We have to push them to the right until they meet the generator σ_k already belonging to the set I—see the figure below



We proceed in two subsequent steps.

(1) We push the generator σ_k located at point A until it meets the generator σ_{k+1} located at point B. The local transition probability of such process is p_1, where

$$p_1 = \frac{2n-6}{2n-1}. \tag{21}$$

It easy to understand that p_1 is the probability to commute a given generator inside the reduced word with its right neighbour.

(2) Completing the first process we push the pair $\sigma_k\sigma_{k+1}$ until it meets the generator σ_k located at the point C. The local transition probability of such a process is $(p_1\, p_2)$, where

$$p_2 = \frac{2n-2}{2n-1}. \tag{22}$$

p_2 is the conditional probability to commute σ_k *under the condition* that σ_{k+1} commutes as well.

We arrive finally at the equation for $Q(\mu')$:

$$Q(\mu') = \left(\frac{1}{2n}\right)^2 \left(\sum_{m,m'=0}^{m+m'\leqslant\mu} p_1^m (p_1\, p_2)^{m'}\right). \tag{23}$$

The answer for $Q(\mu')$ in the limit $n \gg 1$ reads:

$$Q(\mu') = \tfrac{1}{30} - \tfrac{1}{5}\mathrm{e}^{-\frac{5}{2}\mu'} + \tfrac{1}{6}\mathrm{e}^{-3\mu'}. \tag{24}$$

Moreover, for given $\sigma_k \in I$, not only can the sequence $\sigma_k\sigma_{k+1}\sigma_k$ be used for braiding relations but also 5 other sequences (namely, $\sigma_k^{-1}\sigma_{k+1}\sigma_k$, $\sigma_k^{\pm1}\sigma_{k-1}\sigma_k$, $\sigma_k^{-1}\sigma_{k+1}^{-1}\sigma_k$, $\sigma_k^{-1}\sigma_{k-1}^{-1}\sigma_k$).

Thus, in equation (13), η' has to be replaced by,

$$\eta''(\mu') = \eta'(\mu')(1 + 6Q(\mu')) \tag{25}$$

while equation (16) remains unchanged. The results are shown in the lowest part of figure 1 (the points are the numerical simulations for B_{n+1}, $n = 100$; the full curve corresponds to equations (25), (16)).

At the end of this section let us mention two important facts.

- For small N' (typically, for $N' < 1$ i.e. $N < n$), we obtain

$$\left(\frac{\langle\mu\rangle}{N}\right)_{\mathcal{LF}_{n+1}} \approx \left(\frac{\langle\mu\rangle}{N}\right)_{B_{n+1}} \tag{26}$$

i.e. the 'braiding' plays practically no role because the words are too short to produce sequences such as $\sigma_k\sigma_{k+1}\sigma_k$.

- In the asymptotic regime $N' \to \infty$ and $\mu' \to \infty$ we obtain

$$\eta' = \frac{1}{3} \qquad \eta'' = 0.4 \qquad \frac{\langle\mu\rangle}{N} = 0.6. \tag{27}$$

We can now appreciate the impact of the braiding relations. The reductions are increased by about 20% (from $\frac{1}{3}$ to 0.4)—see equation (27) and, simultaneously, $\langle\mu\rangle/N$ is decreased by about 10%. So, in that regime, the groups $\mathcal{LF}_{n+1}$ and B_{n+1} do not coincide (even though they give the same order of magnitude for the quantity $\langle\mu\rangle/N$). This is consistent with our conjecture expressed in [10] where we introduced the concept of the locally free group with 'errors' in commutation relations, $\mathcal{LF}_{n+1}^{\mathrm{err}}$. Recall that in [10] the coincidence between the limit behaviour of the irreducible words in $\mathcal{LF}_{n+1}^{\mathrm{err}}$ and B_{n+1} has been reached if we allow 20% of errors in commutation relations.

3. Random walks with correlations on locally free and braid groups

We come back to the locally free group $\mathcal{LF}_{n+1}(2)$ and suppose, now, that the letters are drawn according to the rules described in the section 1.1. Because of the correlations inside of the words, a 'step by step' treatment as in the standard case is practically untractable. However, rescaling arguments allow us to obtain reliable approximate solutions.

The weak correlations (the case 'A'). The effect of correlations 'A' amounts to a change of equations (12)–(13) into,

$$\frac{\mathrm{d}\eta'}{\mathrm{d}N'} = 2\alpha(1 - 3\eta')\frac{\beta - \alpha\eta'}{1 + \beta - \alpha\eta'} \tag{28}$$

and

$$\frac{\mathrm{d}\mu'}{\mathrm{d}N'} = \beta - \alpha\eta' \tag{29}$$

where

$$\alpha = 1 - 2q_A \qquad \beta = \sqrt{\frac{1 - 2q_A}{1 + 2q_A}}. \tag{30}$$

Let us explain where these equations come from.

Using the same line of thought as in section 2.1 and taking into account the normalization condition (2), we obtain

$$\frac{\mathrm{d}\eta'}{\mathrm{d}N'} = (1 - 3\eta')(1 - 2q_A)\left(1 - \frac{\Pi_2}{\Pi_1}\right). \tag{31}$$

and

$$\frac{d\mu'}{dN'} = \Pi_1 - \Pi_2. \tag{32}$$

However, the probabilities, Π_1 and Π_2 corresponding to the 'increase' and 'decrease' processes ($\Pi_1 + \Pi_2 = 1$) must be computed again to take into account the correlations. This is done as follows.

(1) The case 'A' means that we mainly take care of the situation when the next added letter is the same (with the probability q_A) as the previous added one.

Suppose that, at some time, we draw the letter σ_k. At the next time-step we can add σ_k or σ_k^{-1} with the probability† q_A. Continuing such a process we develop a one-dimensional (1D) 'random walk'

$$W_{1D}^A : \{\sigma_k\sigma_k^{-1}\sigma_k\sigma_k\sigma_k^{-1}\sigma_k^{-1}\ldots\}$$

with a mean 'lifetime' $\tau_A = 1 + 2q_A + (2q_A)^2 + \ldots = 1/(1-2q_A)$. This implies the rescaling of the 'time':

$$N' \to (1-2q_A)N'$$

as can be seen in equation (31).

(2) Another contribution to equations (31) and (32) is connected to the 'mean length', $\langle l\rangle$, of the random chain W_{1D}^A discussed above. To clarify what $\langle l\rangle$ is let us consider the following example:

Number of steps	1D chain	The length l	Probability
1	σ_k	1	$(1-2q_A)$
2	$\sigma_k\sigma_k^{-1}$	0	$q_A(1-2q_A)$
2	$\sigma_k\sigma_k$	2	$q_A(1-2q_A)$
3	$\sigma_k\sigma_k^{-1}\sigma_k = \sigma_k$	1	$q_A^2(1-2q_A)$
3	$\sigma_k\sigma_k^{-1}\sigma_k^{-1} = \sigma_k^{-1}$	1	$q_A^2(1-2q_A)$
3	$\sigma_k\sigma_k\sigma_k = \sigma_k^3$	3	$q_A^2(1-2q_A)$
3	$\sigma_k\sigma_k\sigma_k^{-1} = \sigma_k$	1	$q_A^2(1-2q_A)$
...	...	...	... .

The calculation of $\langle l\rangle$ for given q_A and an infinitely long random chain W_{1D}^A leads to the equation

$$\langle l\rangle = \sum_{k=0}^{\infty} a_k q_A^k (1-2q_A)$$

where a_k obeys the recursion relations

$$a_{2k+1} = 2\,a_{2k}$$
$$a_{2k} - 2\,a_{2k-1} = C_{2k}^k.$$

The final answer for $\langle l\rangle$ is,

$$\langle l\rangle = \sum_{k=0}^{\infty} C_{2k}^k q_A^{2k} = \frac{1}{\sqrt{1-4q_A^2}} \tag{33}$$

† We draw σ_k with probability q_A and σ_k^{-1} with probability q_A independent of each other.

$\langle l\rangle$ is produced during the 'lifetime' τ_A. Thus, equation (13) should be rewritten as

$$\frac{1}{1-2q_A}\frac{\mathrm{d}\mu'}{\mathrm{d}N'} = \langle l\rangle - \eta'. \tag{34}$$

This equation enables us to extract the expressions for Π_1 (and, hence, for $\Pi_2 = 1 - \Pi_1$)—see equation (32) and substitute these values in equation (31). Now equations (28) and (29) follow directly from equations (31) and (32).

Let us stress that naive 'time rescaling' in equation (13), i.e. the equation

$$\frac{1}{1-2q_A}\frac{\mathrm{d}\mu'}{\mathrm{d}N'} = 1 - \eta'$$

(i.e. when $\langle l\rangle = 1$) leads to a wrong result.

The comparison with numerical simulations is shown in figure 2. For the group B_{n+1}—the lower part of the figure 2—we used the same recipe as in section 2.3 to obtain the analytic results (dotted curve). In the limit $N' \to \infty$ our computations give the answer (for the group $\mathcal{LF}_{n+1}$):

$$\eta'_\infty = \frac{1}{3} \qquad \left.\frac{\langle\mu\rangle}{N}\right|_\infty = \sqrt{\frac{1-2q_A}{1+2q_A}} - \frac{1}{3}(1-2q_A). \tag{35}$$

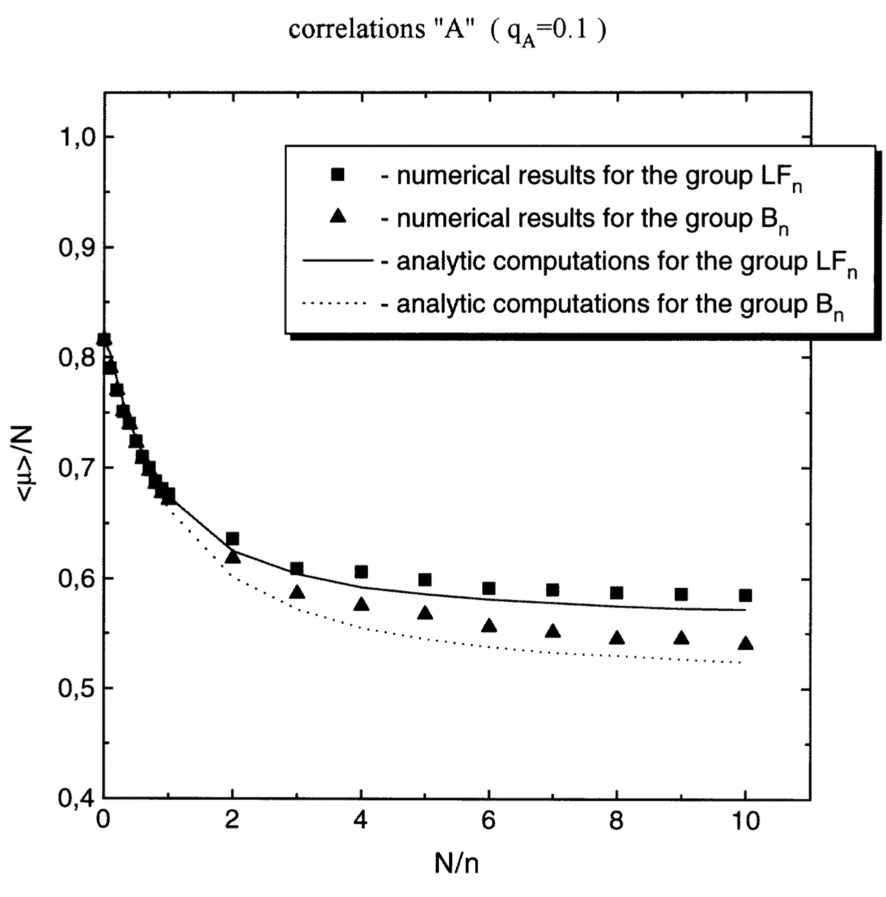


Figure 2. The same as figure 1 except that words are drawn with correlations 'A' ($q_A = 0.1$).

The value $\left.\frac{\langle\mu\rangle}{N}\right|_{\infty}$ is a monotonically decreasing function of q_A when q_A increases from 0 $\left(\frac{\langle\mu\rangle}{N} = \frac{2}{3}\right)$ till its maximal value $\frac{1}{2}\left(\frac{\langle\mu\rangle}{N} = 0\right)$. Clearly, the correlations 'A' enhance the reductions.

Let us pay attention to the fact that

$$\frac{\langle\mu\rangle}{N} \underset{N'\to 0}{\longrightarrow} \sqrt{\frac{1-2q_A}{1+2q_A}} < 1 \qquad \text{if } q_A \neq 0 \tag{36}$$

where $N' \to 0$ means that $n \gg N \gg 1$. This fact is clearly depicted in figure 3. One can see that the agreement between equation (36) (full curve) and numerical simulations for the groups $\mathcal{LF}_{n+1}$ and B_{n+1} (points) is perfect.

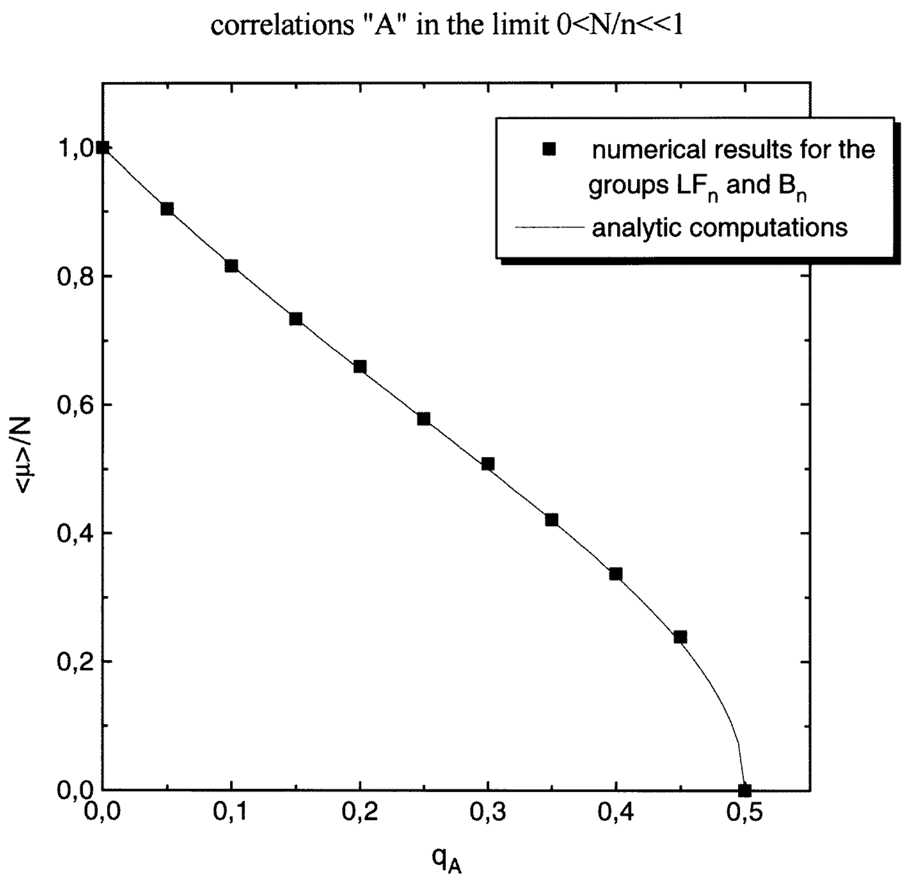


Figure 3. The limit of $\langle\mu\rangle/N$ when $0 < N/n \ll 1$ is plotted as a function of the probability q_A, for locally free and braid groups. Words are drawn with correlations 'A'.

The strong correlations (the case 'B'). The correlations 'B' are dealing with the situation when the next added letter to the initial word is $\sigma_{k\pm 1}^{\pm 1}$ (with the probability q_B) if the previous added one is $\sigma_k^{\pm 1}$—see section 1.1 for the definition and the normalization of probabilities.

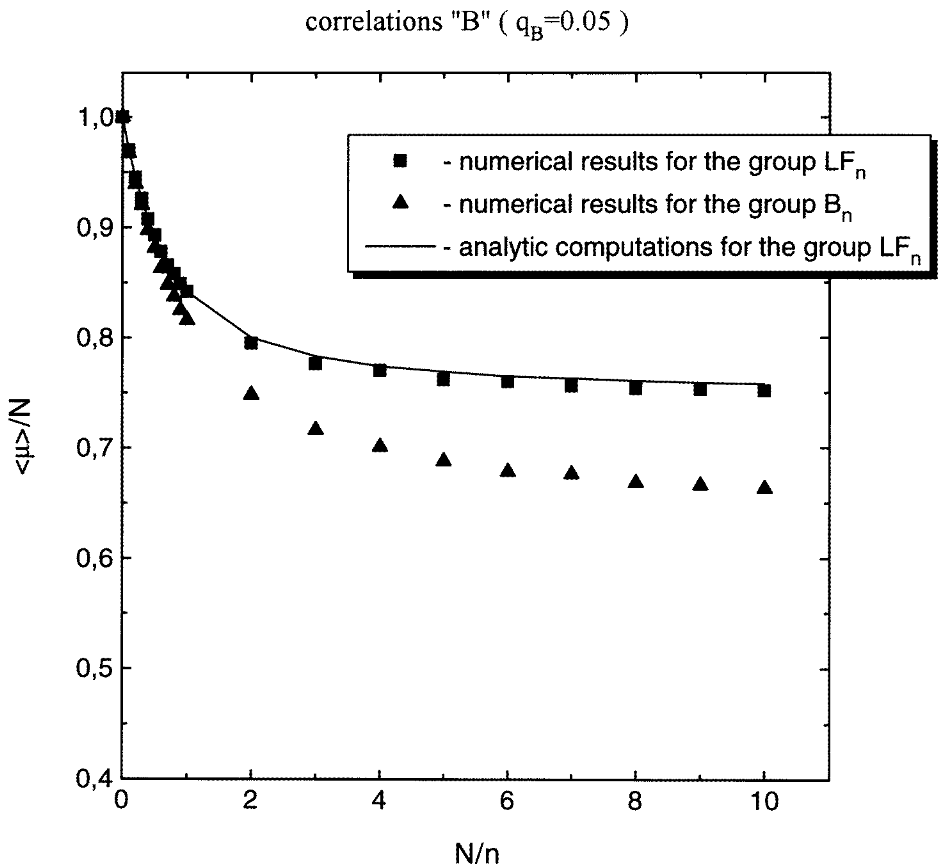


Figure 4. The same as figure 1 except that words are drawn with correlations 'B' ($q_B = 0.05$). (Analytic computations for the group B_n are absent.)

In the spirit of discussion of the case 'A' we can describe our process of successive letter drawing as a development of the 1D Markov chain

$$W_{1D}^{B} : \{\sigma_k \sigma_{k+1} \sigma_k^{-1} \sigma_{k-1}^{-1} \sigma_{k-2} \sigma_{k-3}^{-1} \sigma_{k-2} \ldots\}.$$

The corresponding 'lifetime', τ_B, is $\tau_B = \frac{1}{(1-4q_B)}$.

The chain W_{1D}^{B} can be viewed as a 1D random walk in the 'label space'

$$k \to (k+1) \to k \to (k-1) \to (k-2) \to (k-3) \to (k-2) \ldots$$

with an extension around k of order of $2\sqrt{\tau_B} = \frac{2}{\sqrt{1-4q_B}}$.

Now, if we apply the evolution mechanism of the set I, we immediately realize that all the generators in a zone of extension $\frac{2}{\sqrt{1-4q_B}}$ are erased. (In our example: σ_{k+1} erases σ_k, σ_k^{-1} erases σ_{k+1}, σ_{k-1}^{-1} erases $\sigma_k^{-1}, \ldots$.)

Comparing with the group $\mathcal{LF}_{n+1}(d)$ and following the remark at the beginning of section 2.2, we can define the new effective $d = d_{\rm eff}$ by the equation,

$$2(d_{\rm eff} - 1) = \frac{2}{\sqrt{1-4q_B}}. \tag{37}$$

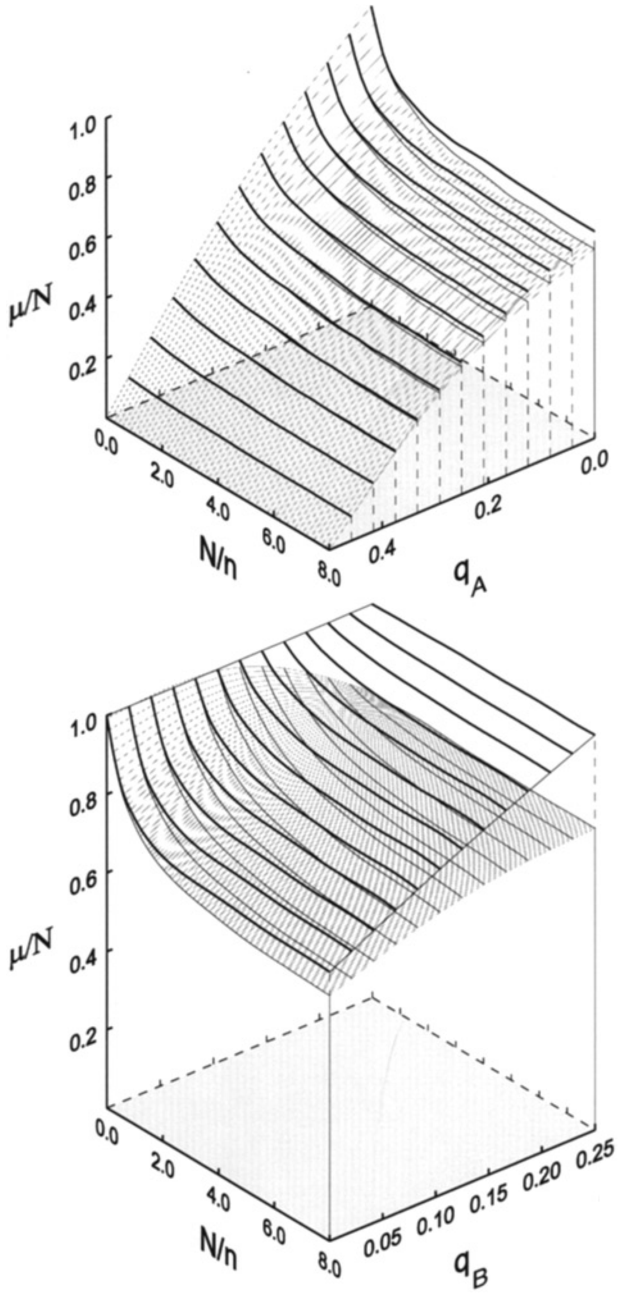


Figure 5. The plot shows the dependence $\mu(N')$ for locally free and braid groups for both kinds of correlations ('A' and 'B'). The corresponding analytic results are available in all cases except for the braid group when words are drawn with correlations 'B'.

Moreover, it is easy to see that the probability Π_2 (reduction process) is equal to

$$\Pi_2 = p_B\eta = np_B\eta' = \left(\frac{1-4q_B}{2}\right)\eta'. \tag{38}$$

We have used the normalization (4) and supposed that $n \gg 1$.

So, we obtain:

$$\frac{d\mu'}{dN'} = \Pi_1 - \Pi_2 = 1 - 2\Pi_2 = 1 - (1 - 4q_B)\eta'. \tag{39}$$

For the evolution of η', we obtain, after time rescaling $N' \to (1 - 4q_B)N'$, the equation

$$\frac{d\eta'}{dN'} = (1 - \eta'(2d_{\text{eff}} - 1))(1 - 4q_B)\left(1 - \frac{\Pi_2}{\Pi_1}\right) \tag{40}$$

that is easily solved. In the limit $N' \to \infty$ we arrive at the following equations

$$\eta'_\infty = \frac{1}{2d_{\text{eff}} - 1} = \frac{1}{\frac{2}{\sqrt{1-4q_B}} + 1} \tag{41}$$

and

$$\left.\frac{\langle\mu\rangle}{N}\right|_\infty = \frac{2 + 4q_B\sqrt{1 - 4q_B}}{2 + \sqrt{1 - 4q_B}}. \tag{42}$$

We can see that $\left.\frac{\langle\mu\rangle}{N}\right|_\infty$ is a monotonically increasing function of q_B from $\frac{2}{3}$ (for $q_B = 0$) until 1 (for the maximal value $q_B = \frac{1}{4}$).

The correlations in the case 'B' increase the length of reduced words in the case of the locally free group $\mathcal{LF}_{n+1}$. The same behaviour is seen numerically for the braid group B_{n+1}.

In figure 4 we compare the results of numerical simulations for the group $\mathcal{LF}_{n+1}(2)$ at $q_B = 0.05$ (dots) with our analytic computations (full curve—solutions of equations (39)–(40)).

Our numerical computations of the normalized reduced word length, μ/N, as a function of normalized initial word length, N/n, are summarized in figure 5. This plot shows the dependence $\mu(N')$ for locally free and braid groups for both kinds of correlations ('A' and 'B'). The corresponding analytic results are available in all cases except for the braid group when words are drawn with correlations 'B'.

4. A ballistic growth model

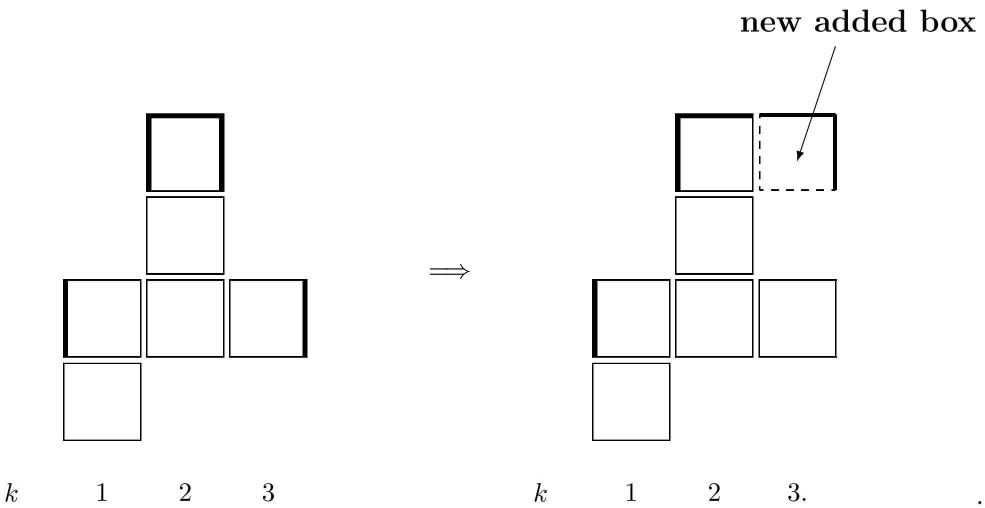
We apply the ideas developed above to the investigation of some statistical properties of a ballistic growth process in $1 + 1$ dimensions.

The standard ballistic deposition can be defined in the following way [12]. Take n columns, of unit width each. A particle, of unit width and height, is dropped vertically in a randomly chosen column and sticks, upon first contact, to the evolving deposit.

Let $h(k, N)$ be the height of the column with the number k ($k \in 1, \ldots, n$) after dropping N particles. The surface of the pile is determined by the function $h(k, N)$. The change of $h(k, N)$ when one extra particle is dropped in column k satisfies the following rule:

$$h(k, N + 1) = \max\{h(k - 1, N), h(k, N) + 1, h(k + 1, N)\}. \tag{43}$$

Schematically this rule corresponds to the following process

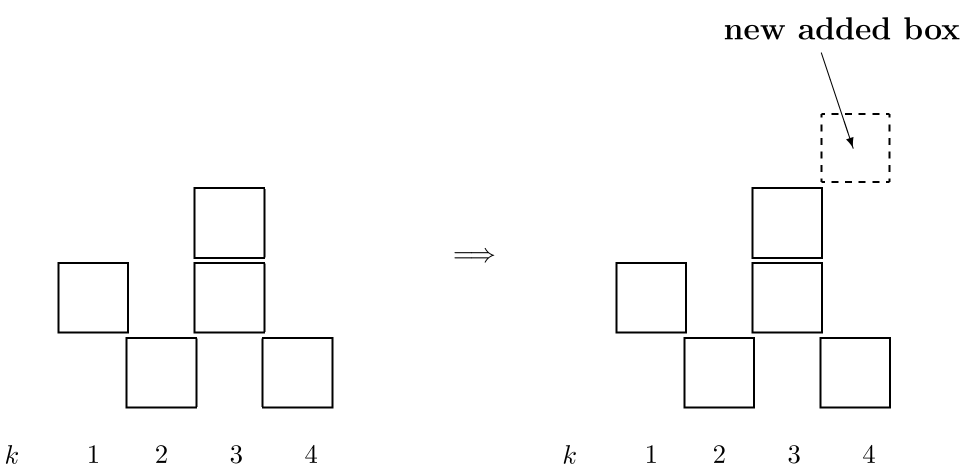


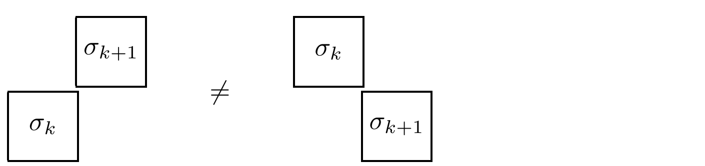
The 'active' box sides (i.e. the sides which can attract the new particles) are shown in boldface.

Let us slightly modify rule (43) and suppose that,

$$h(k, N+1) = \max\{h(k-1, N),\ h(k, N),\ h(k+1, N)\} + 1. \qquad (44)$$

The prescription (44) corresponds to the situation shown below;





It represents the ballistic growth of the pile of unpenetrable particles still of unit height but of width slightly larger than one: two particles dropped in neighbouring columns cannot 'pass through' each other.

In the course of numeric computations we obtain for the average height of the pile, $\overline{h(N')}$, the asymptotic value

$$\frac{\overline{h(N')}}{N'} \approx 4.05$$

for $N' \equiv N/n \gg 1$; while for the standard ballistic model one has

$$\frac{\overline{h(N')}}{N'} \approx 2.13.$$

Thus, the compactness of the pile in our model is about twice smaller.

The collection of peaks and valleys in our model forms a highly rough surface, which develops in the course of particles dropping. We suppose that each 'time-step' corresponds to adding one extra particle to the system. Recall that k is a peak at some time N if $h(k,N) > \max\{h(k-1,N), h(k+1,N)\}$. In what follows we are mainly interested in computing the average number of peaks, $\eta(N)$. As before, we define $\eta' \equiv \eta/n$ and $N' \equiv N/n$.

According to rule (44), two peaks cannot appear in neighbouring columns and we can easily establish the connection with the ideas developed above: a particle dropped in the column k can be viewed as a letter σ_k drawn with the probability $1/n$ over the set $\{\sigma_1, \ldots, \sigma_n\}$ generating the group $\mathcal{LF}_{n+1}(2)$. The 'hardcore' constraint implies the condition $\sigma_i\sigma_j = \sigma_j\sigma_i$ if and only if $i \neq j \pm 1$. Note that we deal in this case with the 'semigroup' $\mathcal{LF}^+_{n+1}$ because we do not use the inverse generators σ_i^{-1} and do not consider the reducing process. From this point of view, our analysis, though analogous, is simpler than for the whole group $\mathcal{LF}_{n+1}$.

Thus, we can easily deduce that *the set of peaks is analogous to the set of reducible letters* I and is reminiscent of the enumeration of 'partially commutative monoids' known in combinatorics [13].

Suppose that two neighbouring peaks are separated by the horizontal interval of length $i \geqslant 1$ and n_i is the number of such intervals. Now we are in a position to write the recursion relations for the process $N' \to N' + 1/n$:

$$\begin{cases} \eta' \to \eta' & \text{occurs with probability } \Pi'_0 = \frac{1}{n}\left(\eta + 2\sum_{i\geqslant 2} n_i\right) \\ \eta' \to \eta' - \dfrac{1}{n} & \text{occurs with probability } \Pi'_- = \frac{n_1}{n} \\ \eta' \to \eta' + \dfrac{1}{n} & \text{occurs with probability } \Pi'_+ = \frac{1}{n}\left(\sum_{i\geqslant 3} n_i\,(i-2)\right) \end{cases} \tag{45}$$

where the conservation condition implies that

$$\Pi'_0 + \Pi'_- + \Pi'_+ = 1.$$

The sum rules (6) and (7) remain unchanged. Comparing equations (45) with equations (8) we find that only Π'_0 differs from Π_0.

In terms of η' and N', we obtain the simple ordinary differential equation for the mean value

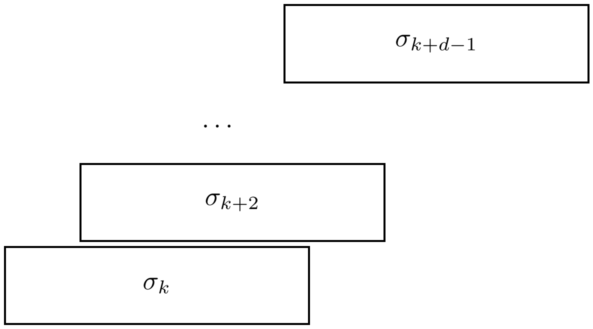
$$\frac{\mathrm{d}\eta'}{\mathrm{d}N'} = 1 - 3\eta'. \tag{46}$$

The solution of equation (46) reads

$$\eta' = \tfrac{1}{3}(1 - \mathrm{e}^{-3N'}). \tag{47}$$

So, asymptotically ($N' \to \infty$), we find that $\frac{1}{3}$ of the columns are peaks.

Let us extend our consideration to the case of unpenetrable particles of widths slightly larger than $(d-1)\times$(unit particle width). It means that now the 'hardcore' condition forces us to consider the generators with the following commutation relations: $\sigma_i\sigma_j = \sigma_j\sigma_i$ if and only if $|i-j| \geqslant d$ (where $d \geqslant 2$). This situation is shown schematically below:



It is a simple matter to see how equations (45)–(47) are changed. We quote the final result only:

$$\eta' = \frac{1}{2d-1}(1 - \mathrm{e}^{-(2d-1)N'}). \tag{48}$$

The case $d = 2$ corresponds to equation (47).

As has been done for the locally free group $\mathcal{LF}_{n+1}(2)$, we are looking at the changes in η' (equation (47)) when we allow the correlations between the subsequently dropped particles (see the section 3 for details).

The weak correlations (the case 'A'). If we draw σ_k at some moment in time, N, then at the next moment in time, $(N+1)$, we have the following situation:

$$\begin{cases} \text{the generator } \sigma_k \text{ appears with the probability } q_A \\ \text{any generator } \sigma_l \ (l \neq k) \text{ appears with the probability } p_A. \end{cases}$$

Due to the absence of inverse generators, the normalization now reads:

$$q_A + (n-1)\, p_A = 1 \tag{49}$$

(cf equation (2)).

The recursion rules for the process $N' \to N' + 1/n$ are now,

$$\begin{cases} \Delta\eta' = 0 & \text{occurs with probability } \Pi'_0 = q_A + (\eta - 1)p_A + 2\sum_{i\geqslant 2} n_i\, p_A \\ \Delta\eta' = -\dfrac{1}{n} & \text{occurs with probability } \Pi'_- = n_1\, p_A \\ \Delta\eta' = +\dfrac{1}{n} & \text{occurs with probability } \Pi'_+ = \sum_{i\geqslant 2} n_i\, (i-2)\, p_A. \end{cases} \tag{50}$$

So, we find

$$\frac{\mathrm{d}\eta'}{\mathrm{d}N'} = (1 - q_A)(1 - 3\eta') \tag{51}$$

and

$$\eta' = \tfrac{1}{3}(1 - \mathrm{e}^{-3(1-q_A)N'}). \tag{52}$$

Asymptotically, again $\frac{1}{3}$ of the columns are peaks and here the effect of correlations leads only to the time rescaling $N' \to (1 - q_A)N'$.

The strong correlations (the case 'B'). If we draw σ_k at some moment in time, N, then at the next moment in time, $(N+1)$, we have the situation,

$$\begin{cases} \text{the generator } \sigma_{k\pm 1} \text{ appears with the probability } q_B \\ \text{any generator } \sigma_l \ (l \neq k \pm 1) \text{ appears with the probability } p_B. \end{cases}$$

The normalization is,

$$2q_B + (n-2)p_B = 1. \tag{53}$$

As has been shown already in section 3 the effect of such a correlation leads to the replacement of $d = 2$ by some effective d_{eff}. According to equation (53), we obtain:

$$d_{\text{eff}} = \frac{1}{\sqrt{1-2q_B}} + 1. \tag{54}$$

In addition, the time must be rescaled as $N' \to (1-2q_B)N'$. Finally we arrive at the following linear differential equation

$$\frac{\mathrm{d}\eta'}{\mathrm{d}N'} = (1-2q_B)(1-(2d_{\text{eff}}-1)\eta') \tag{55}$$

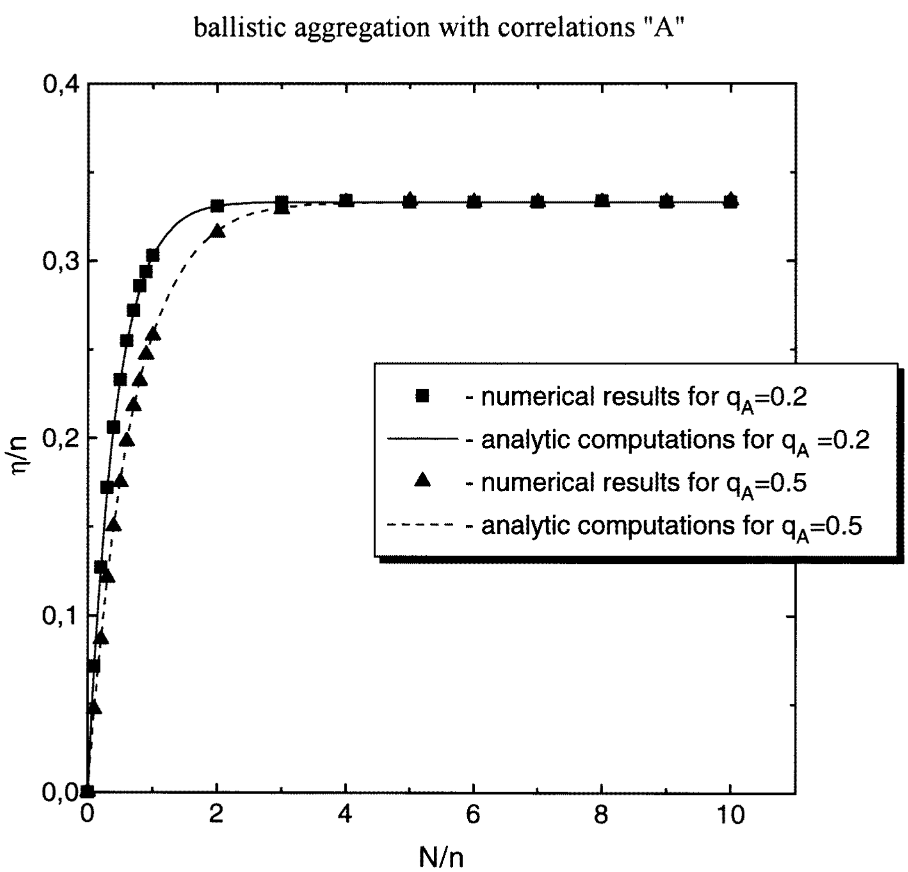


Figure 6. Dependence of normalized amount of the surface peaks, η', on the normalized number of 'pile volume', N', in the ballistic aggregation model. Particles are dropped with correlations 'A'.

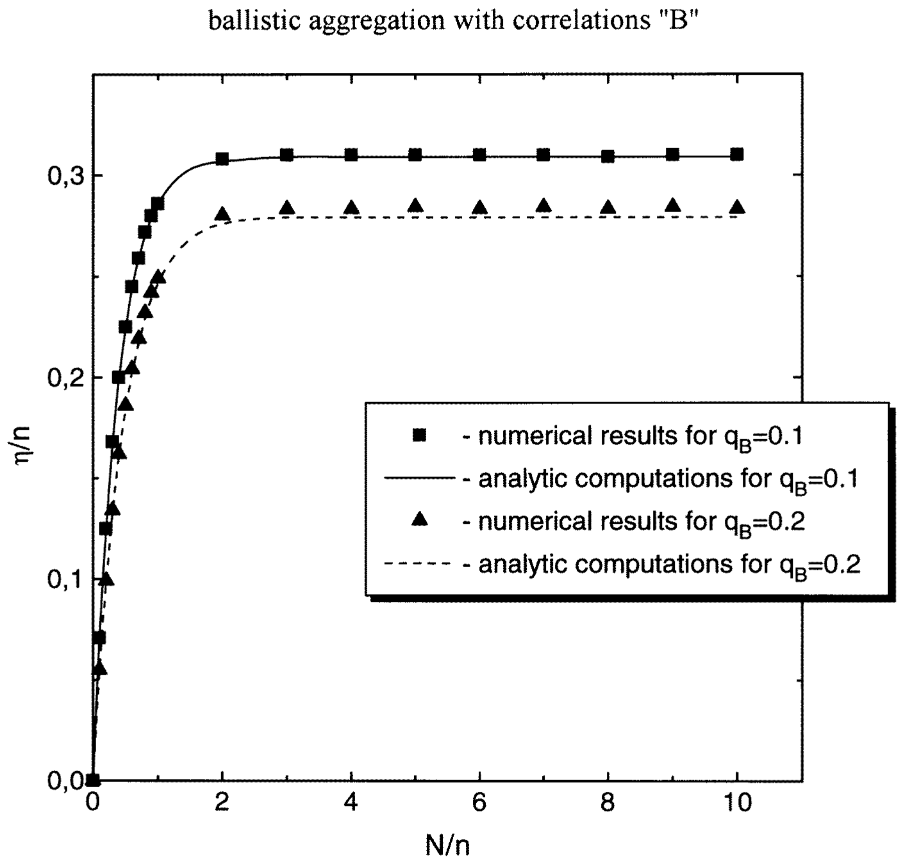


Figure 7. Dependence of normalized amount of the surface peaks, η', on the normalized number of 'pile volume', N', in the ballistic aggregation model. Particles are dropped with correlations 'B'.

which has the solution

$$\eta' = \frac{1}{2d_{\text{eff}} - 1}(1 - e^{-(2d_{\text{eff}}-1)(1-2q_B)N'}). \tag{56}$$

It is worth noting that, in addition to the time rescaling, the correlations 'B' also lead to a change of the asymptotic value of η'.

Comparison of equations (52)–(56) with numerical simulations ($n = 1000$) shows quite good agreement—see figure 6 for correlations of type 'A': (a) $q_A = 0.2$, (b) $q_A = 0.5$ and figure 7 for correlations of type 'B': (a) $q_B = 0.1$, (b) $q_B = 0.2$.

5. Conclusion

The investigation of statistical properties of random walks on braid and locally free groups was undertaken due to the following reasons.

(1) On the basis of performed investigation we constructed the simple mean-field Flory-type theory of interacting braided random walks (bunches of 'directed polymers') in $1 + 1$ dimensions.

(2) The length of the reduced word can be served as a well-defined characteristic of the 'complexity' of knots constructed on the basis of braids. Thus, our study could be regarded as a basis for investigation of the limit behaviour of knot and link topological invariants when the length of the corresponding braid tends to infinity, i.e. when the braid 'grows'.

(3) We believe that the application of the locally free group in the theory of ballistic aggregation could be used: (i) in the consideration of statistical and relaxational properties of 'sandpile models' exhibiting SOC-behaviour (self-organized criticality); (ii) in the microscopic description of the surface growth phenomena.

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